MATHCOUNTS®

2012

■ **State Competition** ■
Countdown Round
Problems 1–80

This section contains problems to be used in the Countdown Round.

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03-S12CDR

1. _____ If 6 and x have the same mean as the set $\{2, 4, 24\}$, what is the value of x?

2. _____ Right $\triangle XYZ$ has sides with lengths x, y and z units, such that $x > y > z$. If $P = 15x^2 + 5y^2 + 5z^2$ and $Q = y^2 + z^2$, what is the value of $\dfrac{P}{Q}$?

3. _____ For all subsets of five distinct integers from the set of consecutive integers $\{1, 2, 3, ..., 99, 100\}$, what is the largest possible difference between the mean and median of a subset?

4. _____ What is the value of the product $\dfrac{1}{5} \times 0.1 \times \dfrac{1}{2} \times 5 \times \dfrac{1}{2} \times 8 \times 5$?

5. _____ (codes) Each widget produced by the Widget Company is to be assigned a unique product code consisting of a letter followed by two digits. What is the number of all possible product codes? Two examples to include are G55 and R92.

6. _____ (cm) The area of a regular hexagon is $96\sqrt{3}$ cm². What is its perimeter, in centimeters?

7. _____ If $n = \sqrt{27}$, what is the value of $n^{\frac{2}{3}}$?

8. _____ If $a \, \Upsilon \, b = b^a \cdot a^{(b-a)}$, what is the value of $(2 \, \Upsilon \, 5)$?

9. _____ (units²) What is the area, in square units, of the region in the xy-plane bounded by $y = x$, $y = 2$ and $x = 6$?

10. _____ If $f(x) = x^2 - 2x - 5$ and $g(x) = x^3 - 11$, what is the value of $f(4) + g(5)$?

11. _____ What is the units digit of the product $2^{42} \cdot 3^{19}$?

12. _____ The prime factorization of 720 is $2^a \cdot 3^b \cdot 5^c$. What is the value of $a + b + c$?

13. _____ (units²) In right $\triangle ABC$, shown here, $\overline{AB} = 15$ units, $\overline{AC} = 24$ units and points D, E and F are the midpoints of $\overline{AC}$, $\overline{AB}$ and $\overline{BC}$, respectively. In square units, what is the area of $\triangle DEF$?
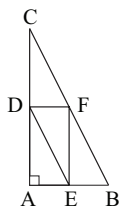
14. _____ In his last game of bowling, Roberto earned a score of 241, and his average score increased from 205 to 209. What score would he have needed in the last game to increase his average to 211?

15. _____ What digit is in the hundred-thousandths place when $(0.5)^5$ is simplified and written as a decimal?

16. _____ (objects) If 110 objects are separated into two groups so that one group contains 150% of the number of objects in the other group, how many objects are in the larger group?

17. _____ (years) In a group of five friends, the sums of the ages of each group of four of them is 58, 59, 61, 62 and 64. What is the age of the oldest of the five friends?

18. _____ In square PQRS, point T is the midpoint of segment QR and $\overline{TU} \perp \overline{SQ}$. What is the ratio of the area of $\triangle QUT$ to the area of square PQRS? Express your answer as a common fraction.
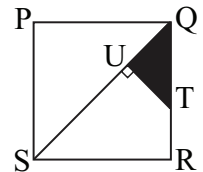
19. _____ If $16 \times 2^x = 2^y + 2^y + 2^y + 2^y$, what is the positive difference between x and y?

20. _____ When two vertices of a convex octagon are selected at random, what is the probability that the segment connecting those two points is a diagonal? Express your answer as a common fraction.

21. _____ What is the sum of all of the positive divisors of 50?

22. _____ If A is $(0, y)$, B is $(4, 4)$ and the length of $\overline{AB}$ is 5 units, what is the sum of all possible values of y?

23. _____ Two prime numbers have a sum of 50. What is the maximum possible value of the product of two such primes?

24. _____ If 30% of x is 12,345, what is 40% of x?

25. _____ The lines $y = -0.4x + 9$ and $y = mx - 2$ are perpendicular. What is the value of m? Express your answer as a decimal to the nearest tenth.

26. _____ (weights) Spherical weights, each with a volume of 0.003 m³, will be dropped into a rectangular tank half-filled with water. The length, width and height of the tank are 6 m, 4 m and 2 m, respectively. How many weights must be dropped into the tank in order for the water level to rise 0.5 m?

27. _____ What is the sum of the first 10 terms of the arithmetic sequence that begins −12, −8, −4, …?

28. _____ (arrange-ments) How many distinct arrangements of four different bowls and one cup can be made if the five items are to be in a line on a shelf and three bowls cannot be next to each other?

29. _____ (triangles) How many non-congruent isosceles triangles have at least one angle of measure 70° and at least one side of length 70 units?

30. _____ The set S contains 19 different integers with a mean of 101. The mean of the ten smallest integers in S is 50, and the mean of the ten largest integers in S is 150. What is the median of all the integers in S?

31. (Number) _____ Eight swimmers, numbered from 1 to 8, stand in ascending order clockwise around the edge of a circular swimming pool. Going around clockwise, every second swimmer still standing will jump into the pool. Swimmer #2 is the first to jump into the pool. What is the number of the swimmer who is the seventh to jump into the pool?

32. _____ What is the smallest of four consecutive prime numbers whose sum is 220?

33. _____ When $3x + 5y$ is subtracted from $x + 7y$, the result is $3x$. What is the ratio of x to y? Express your answer as a common fraction.

34. _____ (cm²) Square ABCD has sides of length 22 cm. What is the area, in square centimeters, of the region shaded gray?
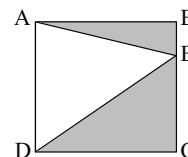
35. _____ Katie writes down six different prime numbers, p, q, r, s, t and u, each less than 20. If $p + q = r + s = t + u$, what is the sum of all the prime numbers less than 20 that Katie did not write down?

36. _____ What is the value of the product $\left(1-\dfrac{11}{15}\right)\left(1-\dfrac{1}{4}\right)$? Express your answer as a common fraction.

37. _____ Six numbers have a mean of 12. Twelve numbers have a mean of 6. What is the mean of the 18 numbers when both sets of numbers are combined?

38. _____ Segments AB and PQ share a midpoint. Point A is at (3, 2) and the endpoints of segment PQ are P(7, 8) and Q(9, 6). What is the sum of the coordinates of point B?

39. _____ The sum of three positive integers is 2012. The sum of the least and greatest of the three integers is 1350, and the sum of the two greatest of these three integers is 1362. What is the positive difference between the greatest integer and the least integer?

40. _____(integers)_ How many positive 3-digit integers contain both even and odd digits?

41. _____(degrees)_ In the figure shown, ABCD is a square, ΔCDE is equilateral and the two shapes are coplanar. What is the measure of ∠CBE, in degrees?
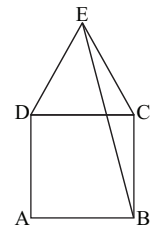
42. _____ What is the value of $3 + 6 - 9 + 12 + 15 - 18 + 21 + 24 - 27 + \ldots + 84 + 87 - 90$?

43. _____ The product of the positive integers a, b and c is 72, and $a > b > c > 1$. What is the greatest possible value of a?

44. _____(degrees)_ What is the degree measure of the indicated exterior angle formed when two regular dodecagons share a common side and two vertices but no common interior points, as shown?
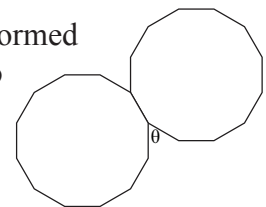
45. _____ If $2^x \cdot 4^{2x} = \sqrt{2^3} \cdot \sqrt{4^2} \cdot \sqrt{8}$, what is the value of x?

46. _____ Given that 10^k is a factor of 25!, what is the greatest possible value of k?

47. _____(locations)_ Inside a square with sides of length 5 units, how many locations are possible for a point P whose distances from each of the four sides are 1 unit, 2 units, 3 units and 4 units? One such location to include is shown.
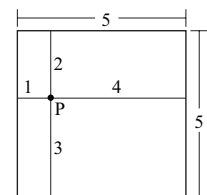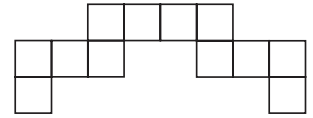
48. _____

3		7
13		
x		15

There are to be nine positive odd integers in the grid shown. Each number is unique, and the numbers in each row, column and diagonal have the same sum. What is the value of x?

49. _____ (units)   What is the perimeter, in units, of this figure composed of 12 unit squares?

50. (___ , ___)   What are the coordinates of the image of $(5, -2)$ after a translation of 3 units right and 7 units down? Express your answer as an ordered pair.

51. _____ (percent)   What percent of the prime numbers less than or equal to 30 are factors of 30?

52. _____ (days)   If 3 brick masons can complete the front of one house in 2 days, how many days will it take a dozen brick masons, working at the same rate, to complete the fronts of 50 identical houses?

53. _____ (Day)   The amount of algae covering the Smith's backyard pond doubled every day until it was completely covered in algae on day 30 of the month. On what day of that month was 75% of the pond algae-free?

54. _____   If $b(b^4 \cdot b^3)^2 = b^{3x}$, what is the value of x?

55. _____   From the set $\{1, 2, 3, 4, 5\}$, Doug will choose three distinct numbers and Regina will choose three distinct numbers. What is the probability that Doug and Regina will choose exactly two of the same numbers as each other? Express your answer as a common fraction.

56. _____ (chips)   Marissa gives $\frac{1}{5}$ of her potato chips to Craig. She then gives $\frac{1}{2}$ of the remaining chips to Pat and then gives $\frac{1}{4}$ of the chips that remain to Zeke. If Marissa now has 12 potato chips left, how many chips did she have before giving any away?

57. _____   The line $y = \frac{3}{4}x + 2$ is reflected over the y-axis, and the resulting line is then reflected over the x-axis. What is the slope of this third line? Express your answer as a common fraction.

58. _____   What is the 101st odd positive integer?

59. _____ In a box containing a dozen cookies there are 4 chocolate chip cookies, 4 oatmeal raisin cookies and 4 macadamia nut cookies. If 3 cookies are chosen at random, without replacement, what is the probability that one cookie of each type is selected? Express your answer as a common fraction.

60. _____ An operation $\otimes$ is defined as $a \otimes b = \dfrac{3(a^2 + b^2)}{ab}$. Find $a \otimes a^2$ if $a = -3$.

61. _____ What is the 6th term of the geometric sequence with a first term of 3 and a second term of -6?

62. _____ For the set $\{2, 5, x, 9, 16\}$, written in ascending order, the mean and median are equal. What is the value of x?

63. _____ What is the smallest positive integer with exactly eight positive factors?

64. _____ Alexio has nine cards: three are red on both sides, three are blue on both sides, and the other three are red on one side and blue on the other side. A card is chosen at random, and the color of one side is observed. What is the probability that the other side of the chosen card is the same color? Express your answer as a common fraction.

65. _____ The square of a positive integer is tripled. When the result is divided by five the quotient is 15. What was the original positive integer?

66. _____ (line segments) How many different straight line segments containing exactly six dots could be drawn in this square array of 36 dots?
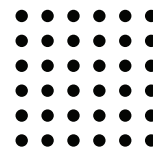
67. _____ What is the sum of all values of x satisfying $|x + 1| = 7$?

68. _____ (units) The area of triangle ABC, shown here, is $36a^8b^4$ units^2, and its height, h, is $8a^2b^2$ units. In units, what is the length of side AC, in terms of a and b?
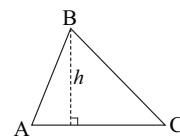
69. _____ Two points, A and B, are randomly selected from the interior of a square. If point A and point B are not the same distance from the intersection of the diagonals, what is the probability that point A is closer to the intersection than point B? Express your answer as a common fraction.

70. _____ The product of three prime numbers is 285. What is their sum?

71. _____ (dollars) The Glee Club is selling raffle tickets for $2 each, or 3 tickets for $5. The prize is a big screen television, which they purchased for $500. If they sell 700 tickets, what is the smallest profit they will make, in dollars?

72. _____ (cm) A worm crawls from point A, due north along a straight path at a rate of 3 cm per second. A minute later, an ant leaves from point A, crawling due south at 5 cm per second. Five minutes after the worm's departure how many centimeters apart from each other are the worm and the ant?

73. _____ In one toss of two fair dice what is the probability that the product of the two numbers rolled is a prime number? Express your answer as a common fraction.

74. _____ What positive integer is closest to the difference $\sqrt{98} - \sqrt{65}$?

75. _____ (percent) What percent of 500 is 4? Express your answer to the nearest tenth.

76. _____ (times) If each edge of a cube is tripled in length to produce a larger cube, the larger cube's surface area is how many times the original cube's surface area?

77. _____ (questions) Malcolm got 50% of the questions on a test correct. If he had 10 questions correct out of the first 12, and $\frac{1}{4}$ of the remaining questions correct, how many questions were on the test?

78. _____ Twin primes are prime pairs that differ by 2. What is the sum of the largest pair of twin primes less than 50?

79. _____ In trapezoid ABCD, shown here, the number of units in the perimeter is between two consecutive, positive integers. What is the sum of those two integers?

80. _____ What is the value of $\dfrac{(2014)^2 - (2006)^2}{(2011)^2 - (2009)^2}$?