HMMT NOVEMBER 2014, 15 NOVEMBER 2014 — GUTS ROUND

Organization ____________________ Team ____________________ Team ID# __________

1. [**5**] Solve for x in the equation $20 \cdot 14 + x = 20 + 14 \cdot x$.

2. [**5**] Find the area of a triangle with side lengths 14, 48, and 50.

3. [**5**] Victoria wants to order at least 550 donuts from Dunkin' Donuts for the HMMT 2014 November contest. However, donuts only come in multiples of twelve. Assuming every twelve donuts cost \$7.49, what is the minimum amount Victoria needs to pay, in dollars? (Because HMMT is affiliated with MIT, the purchase is tax exempt. Moreover, because of the size of the order, there is no delivery fee.)

HMMT NOVEMBER 2014, 15 NOVEMBER 2014 — GUTS ROUND

Organization ____________________ Team ____________________ Team ID# __________

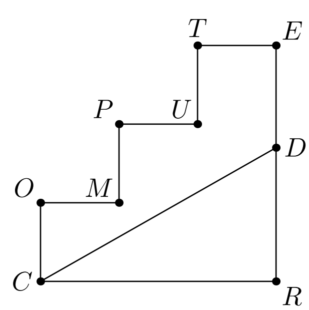
4. [**6**] How many two-digit prime numbers have the property that both digits are also primes?

5. [**6**] Suppose that x, y, z are real numbers such that

$$x = y + z + 2, \quad y = z + x + 1, \quad \text{and} \quad z = x + y + 4.$$

Compute $x + y + z$.

6. [**6**] In the octagon $COMPUTER$ exhibited below, all interior angles are either 90° or 270° and we have $CO = OM = MP = PU = UT = TE = 1$.



Point D (not to scale in the diagram) is selected on segment RE so that polygons $COMPUTED$ and CDR have the same area. Find DR.

Organization ____________________________ Team ____________________________ Team ID# ___________

7. [**7**] Let $ABCD$ be a quadrilateral inscribed in a circle with diameter $\overline{AD}$. If $AB = 5$, $AC = 6$, and $BD = 7$, find CD.

8. [**7**] Find the number of digits in the decimal representation of 2^{41}.

9. [**7**] Let f be a function from the nonnegative integers to the positive reals such that $f(x+y) = f(x) \cdot f(y)$ holds for all nonnegative integers x and y. If $f(19) = 524288k$, find $f(4)$ in terms of k.

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HMMT NOVEMBER 2014, 15 NOVEMBER 2014 — GUTS ROUND

Organization ____________________________ Team ____________________________ Team ID# ___________

10. [**8**] Let ABC be a triangle with $CA = CB = 5$ and $AB = 8$. A circle ω is drawn such that the interior of triangle ABC is completely contained in the interior of ω. Find the smallest possible area of ω.

11. [**8**] How many integers n in the set $\{4, 9, 14, 19, \ldots, 2014\}$ have the property that the sum of the decimal digits of n is even?

12. [**8**] Sindy writes down the positive integers less than 200 in increasing order, but skips the multiples of 10. She then alternately places $+$ and $-$ signs before each of the integers, yielding an expression $+1-2+3-4+5-6+7-8+9-11+12-\cdots-199$. What is the value of the resulting expression?

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HMMT NOVEMBER 2014, 15 NOVEMBER 2014 — GUTS ROUND

Organization ____________________________ Team ____________________________ Team ID# ___________

13. [**9**] Let ABC be a triangle with $AB = AC = \frac{25}{14}BC$. Let M denote the midpoint of $\overline{BC}$ and let X and Y denote the projections of M onto $\overline{AB}$ and $\overline{AC}$, respectively. If the areas of triangle ABC and quadrilateral $AXMY$ are both positive integers, find the minimum possible sum of these areas.

14. [**9**] How many ways can the eight vertices of a three-dimensional cube be colored red and blue such that no two points connected by an edge are both red? Rotations and reflections of a given coloring are considered distinct.

15. [**9**] Carl is on a vertex of a regular pentagon. Every minute, he randomly selects an adjacent vertex (each with probability $\frac{1}{2}$) and walks along the edge to it. What is the probability that after 10 minutes, he ends up where he had started?

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HMMT NOVEMBER 2014, 15 NOVEMBER 2014 — GUTS ROUND

Organization ____________________________ Team ____________________________ Team ID# ___________

16. [**10**] A particular coin has a $\frac{1}{3}$ chance of landing on heads (H), $\frac{1}{3}$ chance of landing on tails (T), and $\frac{1}{3}$ chance of landing vertically in the middle (M). When continuously flipping this coin, what is the probability of observing the continuous sequence HMMT before HMT?

17. [**10**] Let ABC be a triangle with $AB = AC = 5$ and $BC = 6$. Denote by ω the circumcircle of ABC. We draw a circle Ω which is externally tangent to ω as well as to the lines AB and AC (such a circle is called an *A-mixtilinear excircle*). Find the radius of Ω.

18. [**10**] For any positive integer x, define $\text{Accident}(x)$ to be the set of ordered pairs (s,t) with $s \in \{0,2,4,5,7,9,11\}$ and $t \in \{1,3,6,8,10\}$ such that $x+s-t$ is divisible by 12. For any nonnegative integer i, let a_i denote the number of $x \in \{0,1,\ldots,11\}$ for which $|\text{Accident}(x)| = i$. Find

$$a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2.$$

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HMMT NOVEMBER 2014, 15 NOVEMBER 2014 — GUTS ROUND

Organization ____________________________ Team ____________________________ Team ID# ___________

19. [**11**] Let a sequence $\{a_n\}_{n=0}^{\infty}$ be defined by $a_0 = \sqrt{2}, a_1 = 2$, and $a_{n+1} = a_n a_{n-1}^2$ for $n \geq 1$. The sequence of remainders when $a_0, a_1, a_2, \cdots$ are divided by 2014 is eventually periodic with some minimal period p (meaning that $a_m = a_{m+p}$ for all sufficiently large integers m, and p is the smallest such positive integer). Find p.

20. [**11**] Determine the number of sequences of sets $S_1, S_2, \ldots, S_{999}$ such that

$$S_1 \subseteq S_2 \subseteq \cdots \subseteq S_{999} \subseteq \{1,2,\ldots,999\}.$$

Here $A \subseteq B$ means that all elements of A are also elements of B.

21. [**11**] If you flip a fair coin 1000 times, what is the expected value of the product of the number of heads and the number of tails?

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Organization ____________________________ Team ____________________________ Team ID# __________

22. [**12**] Evaluate the infinite sum

$$\sum_{n=2}^{\infty} \log_2\left(\frac{1-\frac{1}{n}}{1-\frac{1}{n+1}}\right).$$

23. [**12**] Seven little children sit in a circle. The teacher distributes pieces of candy to the children in such a way that the following conditions hold.

    - Every little child gets at least one piece of candy.
    - No two little children have the same number of pieces of candy.
    - The numbers of candy pieces given to any two adjacent little children have a common factor other than 1.
    - There is no prime dividing every little child's number of candy pieces.

    What is the smallest number of pieces of candy that the teacher must have ready for the little children?

24. [**12**] Let ABC be a triangle with $AB = 13$, $BC = 14$, and $CA = 15$. We construct isosceles right triangle ACD with $\angle ADC = 90^\circ$, where D, B are on the same side of line AC, and let lines AD and CB meet at F. Similarly, we construct isosceles right triangle BCE with $\angle BEC = 90^\circ$, where E, A are on the same side of line BC, and let lines BE and CA meet at G. Find $\cos\angle AGF$.

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HMMT NOVEMBER 2014, 15 NOVEMBER 2014 — GUTS ROUND

Organization ____________________________ Team ____________________________ Team ID# __________

25. [**13**] What is the smallest positive integer n which cannot be written in any of the following forms?

    - $n = 1 + 2 + \cdots + k$ for a positive integer k.
    - $n = p^k$ for a prime number p and integer k.
    - $n = p + 1$ for a prime number p.
    - $n = pq$ for some distinct prime numbers p and q

26. [**13**] Consider a permutation $(a_1, a_2, a_3, a_4, a_5)$ of $\{1, 2, 3, 4, 5\}$. We say the tuple $(a_1, a_2, a_3, a_4, a_5)$ is *flawless* if for all $1 \leq i < j < k \leq 5$, the sequence (a_i, a_j, a_k) is *not* an arithmetic progression (in that order). Find the number of flawless 5-tuples.

27. [**13**] In triangle ABC, let the parabola with focus A and directrix BC intersect sides AB and AC at A_1 and A_2, respectively. Similarly, let the parabola with focus B and directrix CA intersect sides BC and BA at B_1 and B_2, respectively. Finally, let the parabola with focus C and directrix AB intersect sides CA and CB at C_1 and C_2, respectively.

    If triangle ABC has sides of length 5, 12, and 13, find the area of the triangle determined by lines A_1C_2, B_1A_2 and C_1B_2.

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Organization ______________________________ Team ______________________________ Team ID# ___________

28. [**15**] Let x be a complex number such that $x + x^{-1}$ is a root of the polynomial $p(t) = t^3 + t^2 - 2t - 1$. Find all possible values of $x^7 + x^{-7}$.

29. [**15**] Let ω be a fixed circle with radius 1, and let BC be a fixed chord of ω such that $BC = 1$. The locus of the incenter of ABC as A varies along the circumference of ω bounds a region $\mathcal{R}$ in the plane. Find the area of $\mathcal{R}$.

30. [**15**] Suppose we keep rolling a fair 2014-sided die (whose faces are labelled $1, 2, \ldots, 2014$) until we obtain a value less than or equal to the previous roll. Let E be the expected number of times we roll the die. Find the nearest integer to $100E$.

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Organization ______________________________ Team ______________________________ Team ID# ___________

31. [**17**] Flat Albert and his buddy Mike are watching the game on Sunday afternoon. Albert is drinking lemonade from a two-dimensional cup which is an isosceles triangle whose height and base measure 9cm and 6cm; the opening of the cup corresponds to the base, which points upwards. Every minute after the game begins, the following takes place: if n minutes have elapsed, Albert stirs his drink vigorously and takes a sip of height $\frac{1}{n^2}$ cm. Shortly afterwards, while Albert is busy watching the game, Mike adds cranberry juice to the cup until it's once again full in an attempt to create Mike's cranberry lemonade. Albert takes sips precisely every minute, and his first sip is exactly one minute after the game begins.

    After an infinite amount of time, let A denote the amount of cranberry juice that has been poured (in square centimeters). Find the integer nearest $\frac{27}{\pi^2}A$.

32. [**17**] Let $f(x) = x^2 - 2$, and let f^n denote the function f applied n times. Compute the remainder when $f^{24}(18)$ is divided by 89.

33. [**17**] How many ways can you remove one tile from a 2014×2014 grid such that the resulting figure can be tiled by 1×3 and 3×1 rectangles?

    Warning: The next set of three problems will consist of estimation problems.

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Organization __________________________ Team __________________________ Team ID# __________

34. [**20**] Let M denote the number of positive integers which divide 2014!, and let N be the integer closest to $\ln(M)$. Estimate the value of N. If your answer is a positive integer A, your score on this problem will be the larger of 0 and $\left\lfloor 20 - \frac{1}{8}|A - N| \right\rfloor$. Otherwise, your score will be zero.

35. [**20**] Ten points are equally spaced on a circle. A *graph* is a set of segments (possibly empty) drawn between pairs of points, so that every two points are joined by either zero or one segments. Two graphs are considered the same if we can obtain one from the other by rearranging the points.

    Let N denote the number of graphs with the property that for any two points, there exists a path from one to the other among the segments of the graph. Estimate the value of N. If your answer is a positive integer A, your score on this problem will be the larger of 0 and $\lfloor 20 - 5|\ln(A/N)| \rfloor$. Otherwise, your score will be zero.

36. [**20**] Pick a subset of at least four of the following geometric theorems, order them from earliest to latest by publication date, and write down their **labels** (a single capital letter) in that order. If a theorem was discovered multiple times, use the publication date corresponding to the geometer for which the theorem is named.

    C. (**Ceva**) Three cevians AD, BE, CF of a triangle ABC are concurrent if and only if $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = 1$.

    E. (**Euler**) In a triangle ABC with incenter I and circumcenter O, we have $IO^2 = R(R-2r)$, where r is the inradius and R is the circumradius of ABC.

    H. (**Heron**) The area of a triangle ABC is $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$.

    M. (**Menelaus**) If D, E, F lie on lines BC, CA, AB, then they are collinear if and only if $\frac{BD}{DC}\frac{CE}{EA}\frac{AF}{FB} = -1$, where the ratios are directed.

    P. (**Pascal**) Intersections of opposite sides of cyclic hexagons are collinear.

    S. (**Stewart**) Let ABC be a triangle and D a point on BC. Set $m = BD$, $n = CD$, $d = AD$. Then $man + dad = bmb + cnc$.

    V. (**Varignon**) The midpoints of the sides of any quadrilateral are the vertices of a parallelogram.

    If your answer is a list of $4 \le N \le 7$ labels in a correct order, your score will be $(N-2)(N-3)$. Otherwise, your score will be zero.